THREE LECTURES ON SHEAF COHOMOLOGY

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Abstract. Main references: [Har77] and [Gro57].

Contents

1. Nov 3, Definition of sheaf cohomology

Definition 1.1. Let A be a commutative ring. An A-module L is **injective** if it satisfies the following equivalent conditions:

(1) $\mathrm{Hom}(\bullet, L)$ is right exact.

(2) For all injective homomorphism $j : X' \to X$ and for all homomorphism $f' : X' \to L$, there exists a homomorphism $f : X \to L$ such that $f \circ j = f'$.

(3) For all ideals I of A and for all homomorphism $f : I \to L$, there exists $u \in L$ such that $f(a) = au$ for all $a \in I$.

Proof. We only prove (3) $\implies$ (2).

Let $j : X' \hookrightarrow X$ be an injective homomorphism and let $f' : X' \to L$ be a homomorphism. Define the set S of pairs (Y, g) of A-module Y with a homomorphism $g : Y \to L$ such that $j(X') \subset Y \subset X$ and $(g|_{j(X')}) \circ j = f'$. S is partially ordered by defining $(Y, g) \leq (Y', g')$ if $Y \subset Y'$ and $g'|_Y = g$. Then $S \neq \emptyset$ because $(j(X'), f) \in S$. Also any totally ordered subset (Y_α, g_α) of S has an upper bound $(\bigcup\limits_\alpha Y_\alpha, G)$ with $G|_{Y_\alpha} = g_\alpha$. By Zorn's lemma, S has a maximal element (Y_0, g_0).

Next, we want to show that $Y_0 = X$. If $Y_0 \subsetneq X$, take $x \in X' \setminus Y_0$. Define an ideal $I = \{a \in A \mid ax \in Y_0\}$ of A and a homomorphism $h : I \to L$, $h(a) = g_0(ax)$. By (3), there exists $u \in L$ such that $h(a) = g_0(ax) = au$ for all $a \in I$. Let $Y_1 = Y_0 + Ax$ and $g_1 : Y_1 \to L$, $g_1(y + ax) = g_0(y) + au$ for all $y \in Y_0$, $a \in A$. Then $(Y_0, g_0) < (Y_1, g_1)$ and $(Y_1, g_1) \in S$, which contradicts the maximality of (Y_0, g_0). Therefore $Y_0 = X$ and we take $f = g_0$. $\square$

Example 1.2. $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group.

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Proof. Abelian groups are $\mathbb{Z}$-modules. We verify Definition 1.1(3). For zero ideal it is trivial. Let (n) be an nonzero ideal of $\mathbb{Z}$. Let $f : (n) \to \mathbb{Q}/\mathbb{Z}$ be a $\mathbb{Z}$-homomorphism. Suppose $f(n) = q + \mathbb{Z}$ for some $q \in \mathbb{Q} \cap [0, 1)$. Let $u = \frac{1}{n}q + \mathbb{Z}$. Then for all $m \in \mathbb{Z}$, $mn \in I$ and $f(mn) = mq + \mathbb{Z} = mnu$. □

Let L be an A-module. Let $\widehat{L} = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z})$. Let $\widehat{\widehat{L}} = \mathrm{Hom}_{\mathbb{Z}}(\widehat{L}, \mathbb{Q}/\mathbb{Z})$. Then we have a homomorphism $i_L : L \to \widehat{\widehat{L}}$ defined as follows: For all $x \in L$ and $f \in \widehat{L}$, $(i_L(x))(f) = f(x)$.

Lemma 1.3. $i_L : L \to \widehat{\widehat{L}}$ is an injective homomorphism.

Proof. Suppose $x \in L$ and $x \neq 0$. Let $\langle x \rangle$ be the cyclic abelian group generated by x. Then $\langle x \rangle$ is a subgroup of L. When x has order n, we define $g : \langle x \rangle \to \mathbb{Q}/\mathbb{Z}$, $g(x) = \frac{1}{n} + \mathbb{Z}$. When x has infinite order, we define $g : \langle x \rangle \to \mathbb{Q}/\mathbb{Z}$, $g(x) = \frac{1}{2} + \mathbb{Z}$.

By Example 1.2, $\mathbb{Q}/\mathbb{Z}$ is injective, there exists a homomorphism $f : L \to \mathbb{Q}/\mathbb{Z}$ such that $f|_{\langle x \rangle} = g$ and $f(x) = g(x) \neq 0$. □

Lemma 1.4. If L is projective, then $\widehat{L}$ is injective.

Proof. Let $j : X' \to X$ be an injective homomorphism. Let $f : X' \to \widehat{L}$ be a homomorphism. Since $\mathbb{Q}/\mathbb{Z}$ is injective, by Definition 1.1(1), $\widehat{j} : \widehat{X} \to \widehat{X'}$ defined by $\widehat{j}(g) = g \circ j$ is surjective. We also have $\widehat{f} : \widehat{\widehat{L}} \to \widehat{X'}$. Since $\widehat{L}$ is projective, there exists $h : L \to \widehat{X}$ such that $\widehat{j} \circ h = \widehat{f} \circ i_L$.

$$\begin{tikzcd}
& L \arrow[d, "i_L"] \arrow[ddl, "\exists h"'] & \\
& \widehat{\widehat{L}} \arrow[d, "\widehat{f}"] & \\
\widehat{X} \arrow[r, "\widehat{j}"] & \widehat{X'} \arrow[r] & 0
\end{tikzcd}$$

Then we have $\widehat{h} : \widehat{\widehat{X}} \to \widehat{L}$. By Lemma 1.3, $i_X : X \to \widehat{\widehat{X}}$ is an injective homomorphism. Then for all $x' \in X'$ and $l \in L$, we have

$$((\widehat{h} \circ i_X \circ j)(x'))(l) = (i_X(j(x')))(h(l)) = (h(l))(j(x')) = ((\widehat{j} \circ h)(l))(x')$$

$$= ((\widehat{f} \circ i)(l))(x') = (\iota(l))(f(x')) = f(x')(l)$$

Therefore $\widehat{h} \circ i_X \circ j = f$ and hence $\widehat{L}$ is injective.

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[rr, "j"] \arrow[dd, "f"] & & X \arrow[dl, "i_X"] \\
& & \widehat{\widehat{X}} \arrow[dl, "\widehat{h}"] & \\
& \widehat{L} & &
\end{tikzcd}$$

□

Proposition 1.5. Let A be a commutative ring. Any A-module is isomorphic to a submodule of an injective A-module.

Proof. Let L be an A-module. There exists a projective module P with a surjection $\pi : P \to \widehat{L}$. Then, $\widehat{\pi} : \widehat{\widehat{L}} \to \widehat{P}$ is an injection. By Lemma 1.3 we have an injection $\widehat{\pi} \circ i : L \to \widehat{F}$. By Lemma 1.4, $\widehat{F}$ is an injective A-module. □

Let X be a topological space.

Fact 1.6. Let F, G, H be sheaves of abelian groups on X. Then $F \xrightarrow{a} G \xrightarrow{b} H$ is exact iff $F_x \xrightarrow{a_x} G_x \xrightarrow{b_x} H_x$ is exact for all $x \in X$.

Proof. Omit. □

Let $\mathcal{O}_X$ be a sheaf of rings on X. Let $\mathbf{Mod}(\mathcal{O}_X)$ be the category of sheaves of $\mathcal{O}_X$-modules.

Theorem 1.7. $\mathbf{Mod}(\mathcal{O}_X)$ *has enough injectives.*

Proof. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-module.

(1) Construct a sheaf I. Then for all $x \in X$, $\mathscr{F}_x$ is a $\mathcal{O}_{X,x}$-module. By Proposition 1.5, there exists an injective $\mathcal{O}_{X,x}$-module I with an injection $i_x : \mathscr{F}_x \to I_x$. Then I_x is a sheaf on $\{x\}$. Suppose $j_x : \{x\} \to X$ is the inclusion map. Then $(j_x)_* I_x$ is a sheaf of $\mathcal{O}_X$-module. Let $I = \prod_{x\in X} (j_x)_* I_x$. Then I is a sheaf of $\mathcal{O}_X$-module.

(2) Construct an injection $\iota : \mathscr{F} \to I$. Since

$$\mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_X)}(F, I) = \mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_X)}(F, \prod_{x\in X} (j_x)_* I_x)$$

$$\simeq \prod_{x\in X} \mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_X)}(F, (j_x)_* I_x) \simeq \prod_{x\in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x, I_x)$$

There exists a morphism $\iota : \mathscr{F} \to I$ whose morphism on stalks are $(i_x)_{x\in X}$. Since i_x are all injections, by Fact 1.6, i is an injection.

(3) Show that I is an injective sheaf. $\mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_X)}(\bullet, I)$ is formed by three kinds of functors $\bullet_x$, $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\bullet, I_x)$ and $\prod_{x\in X} \bullet$. By Fact 1.6, $\bullet_x$ is exact for all $x \in X$. Since I_x is injective, by Definition 1.1, $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\bullet, I_x)$ is exact for all $x \in X$. Also, the product $\prod_{x\in X} \bullet$ of exact functors is exact. Therefore $\mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_X)}(\bullet, I)$ is exact. By Definition 1.1, I is injective. □

Let $\mathbf{Ab}$ be the category of abelian groups. Let $\mathbf{Ab}(X)$ be the category of sheaves of abelian groups on X.

Corollary 1.8. $\mathbf{Ab}(X)$ has enough injectives.

Proof. Let $\mathcal{O}_X$ be the locally constant sheaf of rings $\underline{\mathbb{Z}}$. Then the result follows from $\mathbf{Ab}(X) = \mathbf{Mod}(\underline{\mathbb{Z}})$ and Theorem 1.7. □

Lemma 1.9. The **global section** functor $\Gamma(X, \bullet) : \mathbf{Ab}(X) \to \mathbf{Ab}$ such that $\Gamma(X, \mathscr{F}) = \mathscr{F}(X)$ is left exact.

Proof. Suppose $0 \to \mathscr{F} \xrightarrow{i} G \xrightarrow{q} H \to 0$ is exact. We want to show that

$$0 \to \mathscr{F}(X) \xrightarrow{i_X} G(X) \xrightarrow{q_X} H(X)$$

is exact.

$\underline{i_X \text{ is injective.}}$ In fact, suppose $s \in \mathscr{F}(X)$ such that $i_X(s) = 0$. Then $i_x(s_x) = (i_X(s))_x = 0$ for all $x \in X$. By Fact 1.6, $s_x = 0$ for all $x \in X$. For each $x \in X$, there exists an open neighborhood U_x of x such that $s|_{U_x} = 0$. Since $(U_x)_{x\in X}$ is an open covering of X and $\mathscr{F}$ is a sheaf, $s = 0$.

$\underline{\ker(q_X) \supset \mathrm{Im}(i_X).}$ In fact, suppose $s \in \mathscr{F}(X)$, let $t = q_X(i_X(s)) \in H(X)$. Then $t_x = q_x(i_x(s_x)) = 0$. For each $x \in X$, there exists an open neighborhood U_x of x such that $t|_{U_x} = 0$. Since $(U_x)_{x\in X}$ is an open covering of X and H is a sheaf, $t = 0$.

$\underline{\ker(q_X) \subset \mathrm{Im}(i_X).}$ In fact, suppose $s \in G(X)$ such that $q_X(s) = 0$. Then $q_x(s_x) = (q_X(s))_x = 0$ for all $x \in X$. By Fact 1.6, $s_x = i_x(t_x)$ for some $t_x \in \mathscr{F}_x$ for all $x \in X$. Then there exists an open neighborhood U_x of x and $t_{U_x} \in \mathscr{F}(U_x)$ such that $s|_{U_x} = i_{U_x}(t_{U_x})$. If $U_x \cap U_y \neq \emptyset$, then $i_{U_x\cap U_y}(t_{U_x}|_{U_x\cap U_y} - t_{U_x}|_{U_x\cap U_y}) = s|_{U_x\cap U_y} - s|_{U_x\cap U_y} = 0$. Since i is injective, $t_{U_x}|_{U_x\cap U_y} = t_{U_x}|_{U_x\cap U_y}$. Since $\mathscr{F}$ is a sheaf and $(U_x)_{x\in X}$ is a covering of X, there exists $t \in \mathscr{F}(X)$ such that $t|_{U_x} = t_{U_x}$. Since $(i_X(t))|_{U_x} = i_{U_x}(t|_{U_x}) = i_{U_x}(t_{U_x}) = s|_{U_x}$ for all $x \in X$, G is a sheaf and $(U_x)_{x\in X}$ is a covering of X, we have $i_X(t) = s$. $\square$

Example 1.10. $\Gamma(X, \bullet) : \mathbf{Ab}(X) \to \mathbf{Ab}$ is not necessarily right exact.

Let $X = \mathbb{C}^*$ with analytic topology. Let $\underline{\mathbb{Z}}$ be the locally constant sheaf associated to $\mathbb{Z}$. Let $\mathcal{O}$ be the sheaf of holomorphic functions. Let $\mathcal{O}^*$ be the sheaf of invertible holomorphic functions. Then we have an exact sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\bullet 2\pi\sqrt{-1}} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 1.$$

For all $w \in \mathbb{C}$, $\underline{\mathbb{Z}}_w = \mathbb{Z}$, $\mathcal{O}_w = \{f : U_w \to \mathbb{C} \mid \exists f'(z), \forall z \in U_w\}$ and $\mathcal{O}^*_w = \{f : U_w \to \mathbb{C} \mid \exists f'(z), \forall z \in U_w, f(w) \neq 0, \}$, where U_w is some open neighborhood of w.

$$0 \to \mathbb{Z} \xrightarrow{\bullet 2\pi\sqrt{-1}} \mathcal{O}_w \xrightarrow{\exp} \mathcal{O}^*_w \to 0$$

is exact at $\mathbb{Z}$ and $\mathcal{O}_w$ because $e^{2\pi\sqrt{-1}} = 0$. And exp is surjective because let $g(z) = \ln(f(z))$ on a neighborhood of w such that ln is a well-defined logarithm function on a band neighborhood of $f(w)$ of width 2π. Then $g'(w) = \dfrac{f'(w)}{f(w)}$ exists and hence g is a inverse image of f.

However, $\exp : \mathcal{O}(\mathbb{C}^*) \to \mathcal{O}^*(\mathbb{C}^*)$ is not surjective because $\mathrm{Id}_{\mathbb{C}^*} \in \mathcal{O}^*(\mathbb{C}^*)$ does not have an inverse image. If not, there exists $f \in \mathcal{O}(\mathbb{C}^*)$ such that $e^f = \mathrm{Id}_{\mathbb{C}^*}$, then

$$f|_{\mathbb{C}^*\setminus(-\infty,0)}(re^{\theta\sqrt{-1}}) = \ln(r) + \theta\sqrt{-1} + 2n\pi\sqrt{-1}$$

for some $n \in \mathbb{Z}$, where $r > 0$ and $-\pi < \theta < \pi$;

$$f|_{\mathbb{C}^*\setminus(0,+\infty)}(re^{\theta'\sqrt{-1}}) = \ln(r) + \theta'\sqrt{-1} + 2m\pi\sqrt{-1}$$

for some $n \in \mathbb{Z}$, where $r > 0$ and $0 < \theta' < 2\pi$.

If $0 < \theta = \theta' < \pi$, then $n = m$. If $-\pi < \theta' = \theta + 2\pi < 0$, then $n = m - 1$. Contradiction.

Definition 1.11. Let X be a topological space (By Corollary 1.8, it has enough injectives). Let $\Gamma(X,\bullet) : \mathbf{Ab}(X) \to \mathbf{Ab}$ be the global section functor (By Lemma 1.9 it is left exact). Define

$$H^i(X,\bullet) = R^i\Gamma(X,\bullet) : \mathbf{Ab}(X) \to \mathbf{Ab},\ i = 0, 1, \dots$$

be the sequence of right derived functors of $\Gamma(X,\bullet)$. Let $\mathscr{F}$ be a sheaf of abelian groups on X. Then $H^i(X,\mathscr{F})$ is called the i-th **sheaf cohomology** of $\mathscr{F}$.

Remark 1.12. (1) Definition 1.11 provides the first way to calculate sheaf cohomology. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-module. Let $0 \to \mathscr{F} \to I^0 \to I^1 \to \cdots$ be a injective resolution of $\mathscr{F}$. Apply $\Gamma(X,\bullet)$ to the resolution, we obtain a complex

$$0 \to \Gamma(X, I^0) \to \Gamma(X, I^1) \to \cdots$$

The usual i-th cohomology of this complex is $h^i(\Gamma(X, I^\bullet)) \simeq H^i(X,\mathscr{F})$.

(2) The disadvantage of (1) is that: Despite that there are enough injectives, there are still "too few". That is why we need flasque sheaves.

Definition 1.13. Let X be a topological space. Let $\mathscr{F}$ be a sheaf of abelian groups on X. We say that $\mathscr{F}$ is **flasque** if for all inclusion of open sets $V \subset U$ in X, the restriction $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

Fact 1.14. Let U be an open set of X. Let $j : U \to X$ be the inclusion map. Let $j_!(\mathscr{F})$ be the sheaf associated to the presheaf

$$V \mapsto \begin{cases} \mathscr{F}(V), & V \subset U; \\ 0, & V \not\subset U \end{cases}$$

Then for all $x \in X$

$$j_!(\mathscr{F})_x = \begin{cases} \mathscr{F}_x, & x \in U; \\ 0, & x \notin U. \end{cases}$$

Proof. Omit. □

Lemma 1.15. Let $\mathcal{O}_X$ be a sheaf of rings on X. Let U be an open set of X with inclusion $j : U \to X$. Write $\mathcal{O}_U = j_!(\mathcal{O}_X|U)$. Then for all sheaf of $\mathcal{O}_X$-modules G,

$$\mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_X)}(\mathcal{O}_U, G) \simeq G(U).$$

Proof. Define $\alpha : \mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_X)}(\mathcal{O}_U, G) \to G(U)$. For all natural transformation $f : \mathcal{O}_U \to G$ and for all open subset $V \subset U$, there exists $f_V : \mathcal{O}_U(V) = \mathcal{O}_X(V) \to G(V)$ commuting with restrictions. In particular, we have $f_U : \mathcal{O}_X(U) \to G(U)$, define $\alpha(f) = f_U(1)$.

Define $\beta : G(U) \to \mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_X)}(\mathcal{O}_U, G)$. For all $g \in G(U)$ and for all open subset $V \subset U$, $g|V \in G(V)$. Define $\beta(g) : \mathcal{O}_U \to G$ such that $\beta(g)_V(x) = x \cdot g|V$ for all $x \in \mathcal{O}_U(V) = \mathcal{O}_X(V)$.

For all $f \in \mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_X)}(\mathcal{O}_U, G)$, V open in U and $x \in \mathcal{O}_U(V)$,

$$\beta(\alpha(f))_V(x) = x \cdot (\alpha(f)|V) = x \cdot f_U(1)|V = x \cdot f_V(1) = f_V(x),$$

Hence $\beta \circ \alpha = \mathrm{Id}$.

Conversely, for all $g \in G(U)$,

$$\alpha(\beta(g)) = \beta(g)_U(1) = 1 \cdot g|U = g,$$

Hence $\alpha \circ \beta = \mathrm{Id}$.

This is basically like the proof of Yoneda lemma. □

The next lemma shows that there are "more" flasque sheaves than injective sheaves.

Lemma 1.16. Let $\mathcal{O}_X$ be a sheaf of rings on X. Any injective $\mathcal{O}_X$-module is flasque.

Proof. Suppose $V \subset U$ is an inclusion of open sets in X. By Fact 1.14, the inclusion gives the canonical injection $\mathcal{O}_V \to \mathcal{O}_U$. Let I be an injective sheaf of $\mathcal{O}_X$-module. By Definition 1.1, there exists a canonical surjection

$$\mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_X)}(\mathcal{O}_U, I) \to \mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_X)}(\mathcal{O}_V, I)$$

By Lemma 1.15, this surjection is identified with the restriction $I(U) \to I(V)$. $\square$

2. Nov 10, Sheaf cohomology and flasque resolution

Review: Let X be a topological space. Let $\mathcal{O}_X$ be a sheaf of rings on X.

(1) $\mathbf{Mod}(\mathcal{O}_X)$ has enough injectives. e.g. $\mathbf{Ab}(X)$.
(2) $\Gamma(X, \bullet) : \mathbf{Ab}(X) \to \mathbf{Ab}$ is left exact. e.g. $0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^* \to 1$ on $\mathbb{C}^*$.
(3) $H^n(X, \bullet) = R^n\Gamma(X, \bullet)$. e.g. $H^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$.
(4) A sheaf $\mathscr{F}$ is flasque if the restriction $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective for all inclusions of open sets $V \subset U$. e.g. injectives sheaves of $\mathcal{O}_X$-modules.

Definition 2.1. A sheaf of $\mathcal{O}_X$-module $\mathscr{F}$ is **acyclic** if $H^n(X, \mathscr{F}) = 0$ for all $n \geq 1$.

Example 2.2. Let I be an injective sheaf of $\mathcal{O}_X$-modules. Then $0 \to I \to I \to 0$ is an injective resolution of I. We obtain a complex $0 \to \Gamma(X, I) \to 0$. Hence $H^n(X, I) = 0$ for all $i \geq 1$. Hence, injective sheaves of $\mathcal{O}_X$-modules are acyclic.

Next we want to show that flasque sheaves of $\mathcal{O}_X$-modules are acyclic. Fix the following notations for Lemma 2.3, Lemma 2.4 and Theorem 2.5. Let $\mathscr{F}$ be a flasque sheaf of $\mathcal{O}_X$-module By Theorem 1.7, there exists an injective sheaf of $\mathcal{O}_X$-module I with an injection $i : \mathscr{F} \to I$. Let $G = \mathrm{coker}(i)$. Since $\mathbf{Mod}(\mathcal{O}_X)$ is an abelian category, G is a sheaf. Then we have an exact sequence

$$0 \to \mathscr{F} \xrightarrow{i} I \xrightarrow{q} G \to 0$$

Lemma 2.3.

$$0 \to \mathscr{F}(U) \xrightarrow{i_U} I(U) \xrightarrow{q_U} G(U) \to 0$$

is exact for all open set U of X.

Proof. By Lemma 1.9, it suffices to show that q_U is a surjection. Suppose $s \in G(U)$. Let $T = \{(V, t) \mid V \subset U,\ t \in I(V),\ q_V(t) = s|_V\}$. Define $(V, t) \leq (V't')$ iff $V \subset V'$ and $t'|_V = t$. First, $T \neq \emptyset$ because $(\emptyset, 0) \in T$. Second, any totally ordered subset (V_α, t_α) has an upper bound $(\bigcup\limits_\alpha V_\alpha, t)$ with $t|_{V_\alpha} = t_\alpha$ as I is a sheaf. By Zorn's lemma, T has a maximal element (V_0, t_0).

We want to show that $V_0 = U$ and hence t_0 is an inverse image of s. If not, then there exists $x \in U \setminus V_0$. Since $q_x : I_x \to G_x$ is surjective, there exists an open neighborhood W of x and $t' \in I(W)$ such that $q_W(t') = s|_W$. Since $q_{W\cap V_0}(t_0|_{W\cap V_0} - t'|_{W\cap V_0}) = s|_{W\cap V_0} - s|_{W\cap V_0} = 0$, there exists $r' \in \mathscr{F}(W \cap V_0)$ such that $i_{W\cap V_0}(r') = t_0|_{W\cap V_0} - t'|_{W\cap V_0}$. Since $\mathscr{F}$ is flasque, the restriction $\mathscr{F}(W) \to \mathscr{F}(W \cap V_0)$ is surjective. Then there exists $r \in \mathscr{F}(W)$ such that $r|_{W\cap V_0} = r'$. Let $t'' = t' + i_W(r)$. Then $t''|_{W\cap V_0} = t'|_{W\cap V_0} + i_W(r)|_{W\cap V_0} = t_0|_{W\cap V_0}$. Since I is a sheaf, t_0 and t'' are glued to $\widetilde{t} \in I(W \cup V_0)$. We have $(V_0, t_0) < (W \cup V_0, \widetilde{t}) \in T$, a contradiction. Therefore $V_0 = U$ and $q_U(t) = s$. $\square$

Lemma 2.4. G is flasque.

Proof. Suppose $V \subset U$ is an inclusion of open sets in X. By Lemma 2.3, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}(U) \arrow[r] \arrow[d, "a"] & I(U) \arrow[r] \arrow[d, "b"] & G(U) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \mathscr{F}(V) \arrow[r] & I(V) \arrow[r] & G(V) \arrow[r] & 0
\end{tikzcd}$$

where a, b, c are restrictions. By Snake lemma, we have an exact sequence

$$\cdots \to \operatorname{coker}(b) \to \operatorname{coker}(c) \to 0.$$

Since I is injective, by Lemma 1.16, it is flasque and hence $\operatorname{coker}(b) = 0$. Therefore $\operatorname{coker}(c) = 0$, i.e. G is flasque. $\square$

Theorem 2.5. *Flasque sheaves of $\mathcal{O}_X$-modules are acyclic.*

Proof. Let $\mathscr{F}$ be a flasque sheaf of $\mathcal{O}_X$-module. We want to show that $H^n(X, \mathscr{F}) = 0$ for all $n \geq 1$.

We use induction on n. Let I, G be as above. Let $I^0 = I$. Let I^1 be an injective $\mathcal{O}_X$-module containing G. Then $0 \to \mathscr{F} \to I^0 \to I^1 \to \cdots$ gives a complex

$$0 \to \Gamma(X, I) \xrightarrow{d^0} \Gamma(X, I^1) \xrightarrow{d^1} \cdots$$

So

$$H^1(U, \mathscr{F}) = \frac{\ker(d^1)}{\operatorname{Im}(d^0)} = \frac{G(U)}{\operatorname{Im}(q_U)} = 0$$

for all open set U of X. In particular, $H^1(X, \mathscr{F}) = 0$.

Suppose the n-th cohomology vanishes for all flasque sheaves. In particular, $H^n(X, \mathscr{F}) = 0$. We need to show that $H^{n+1}(X, \mathscr{F}) = 0$. The short exact sequence $0 \to \mathscr{F} \to I \to G \to 0$ gives a long exact sequence

$$\cdots \to H^n(X, I) \to H^n(X, G) \to H^{n+1}(X, \mathscr{F}) \to H^{n+1}(X, I) \to \cdots$$

Since I is injective, by Example 2.2, $H^{n+1}(X, I) = 0$. By Lemma 2.4, G is flasque. By inductive hypothesis, $H^n(X, G) = 0$. Therefore $H^{n+1}(X, \mathscr{F}) = 0$. $\square$

Fact 2.6 (Horseshoe)**.** Let C be an abelian category with enough injectives. Let $0 \to A_1 \xrightarrow{i} A_2 \xrightarrow{p} A_3 \to 0$ be an exact sequence in C. Let $0 \to A_1 \xrightarrow{a} I_1^0 \xrightarrow{a^0} I_1^1 \xrightarrow{a^1} \cdots$ and $0 \to A_3 \xrightarrow{c} I_3^0 \xrightarrow{c^0} I_3^1 \xrightarrow{c^1} \cdots$ be two injective resolutions. Then there exists an injective resolution $0 \to A_2 \xrightarrow{b} I_2^0 \xrightarrow{b^0} I_2^1 \xrightarrow{b^1} \cdots$ and horizontal morphisms such that the following diagram is commutative with exact columns and **split-exact** rows.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "i"] \arrow[d, "a"] & A_2 \arrow[r, "p"] \arrow[d, "b"] & A_3 \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & I_1^0 \arrow[r, "i^0"] \arrow[d, "a^0"] & I_2^0 \arrow[r, "p^0"] \arrow[d, "b^0"] & I_3^0 \arrow[r] \arrow[d, "c^0"] & 0 \\
0 \arrow[r] & I_1^1 \arrow[r, "i^1"] \arrow[d, "a^1"] & I_2^1 \arrow[r, "p^1"] \arrow[d, "b^1"] & I_3^1 \arrow[r] \arrow[d, "c^1"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Proof. Omit. $\square$

Lemma 2.7. Let $C^\bullet$ be a complex in an abelian category with enough injectives. Then there exists an injective resolution

$$0 \to C^\bullet \to I^{\bullet,0} \to I^{\bullet,1} \to \cdots$$

such that

$$0 \to Z^p(C^\bullet) \to Z^p(I^{\bullet,0}) \to Z^p(I^{\bullet,1}) \to \cdots,$$

$$0 \to B^p(C^\bullet) \to B^p(I^{\bullet,0}) \to B^p(I^{\bullet,1}) \to \cdots,$$

$$0 \to h^p(C^\bullet) \to h^p(I^{\bullet,0}) \to h^p(I^{\bullet,1}) \to \cdots$$

are injective resolutions (i.e. **Cartan-Eilenberg resolution** exists).

Proof. We use Z for kernels, B for images and h for usual cohomologies.

First, we select injective resolutions for $B^p(C^\bullet)$ and $h^p(C^\bullet)$ for all p.

Since $0 \to B^{p-1}(C^\bullet) \to Z^p(C^\bullet) \to h^p(C^\bullet) \to 0$, by Fact 2.6, we obtain an injective resolution for $Z^p(C^\bullet)$ for all p.

Since $0 \to Z^p(C^\bullet) \to C^p \to B^p(C^\bullet) \to 0$, by Fact 2.6, we obtain an injective resolution for C^p for all p. □

Acyclic sheaves of $\mathcal{O}_X$-modules provides a second way to calculate sheaf cohomology.

Theorem 2.8 (De Rham-Weil)**.** *Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-module. Let $0 \to \mathscr{F} \to J^0 \to J^1 \to \cdots$ be a acyclic (e.g. flasque) resolution of $\mathscr{F}$. Apply $\Gamma(X, \bullet)$ to the resolution, we obtain a complex*

$$0 \to \Gamma(X, J^0) \to \Gamma(X, J^1) \to \cdots$$

The usual i-th cohomology of this complex is $h^i(\Gamma(X, J^\bullet)) \simeq H^i(X, \mathscr{F})$.

Proof. By Lemma 2.7, let the following be an injective resolution of $0 \to \mathscr{F} \to J^\bullet$.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & I^1 \arrow[r] \arrow[u] & I^{0,1} \arrow[r] \arrow[u] & I^{1,1} \arrow[r] \arrow[u] & I^{2,1} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & I^0 \arrow[r] \arrow[u] & I^{0,0} \arrow[r] \arrow[u] & I^{1,0} \arrow[r] \arrow[u] & I^{2,0} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & F \arrow[r] \arrow[u] & J^0 \arrow[r] \arrow[u] & J^1 \arrow[r] \arrow[u] & J^2 \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

By Lemma 1.9, $\Gamma(X, \bullet)$ is left exact, apply it everywhere, and replace the left column and the bottom row with 0, then we have a double complex $C^{p,q} = \Gamma(X, I^{p,q})$ in **Ab**.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Gamma(X, I^{0,1}) \arrow[r] \arrow[u] & \Gamma(X, I^{1,1}) \arrow[r] \arrow[u] & \Gamma(X, I^{2,1}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \Gamma(X, I^{0,0}) \arrow[r] \arrow[u] & \Gamma(X, I^{1,0}) \arrow[r] \arrow[u] & \Gamma(X, I^{2,0}) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{2.9}$$

First, we take the vertical spectral sequence. Let ${}_vE_0^{p,q} = C^{p,q}$. Since J^p are all acyclic, we have

$$ {}_vE_1^{p,q} = \begin{cases} \Gamma(X, J^p), & q = 0; \\ 0, & q > 0. \end{cases} $$

Then

$$ {}_vE_2^{p,q} = \begin{cases} h^p(\Gamma(X, J^\bullet)), & q = 0; \\ 0, & q > 0. \end{cases} $$

Hence ${}_vE_2^{p,q} = E_\infty^{p,q}$ and ${}_vE_2^{p,q} \Rightarrow h^{p+q}(\Gamma(X, J^\bullet))$.

Next, we take the horizontal spectral sequence. Let ${}_hE_0^{p,q} = C^{q,p}$. Since $0 \to F \xrightarrow{d^{-1}} J^0 \xrightarrow{d^0} J^1 \to \cdots$ is exact, we have $Z^p = B^{p-1}$ for all $p \geq 0$. Let $0 \to Z^p \to Z^{p,0} \to Z^{p,1} \to \cdots$ be an injective resolution of Z^p. By the construction of Lemma 2.7, $I^p = Z^{0,p}$, $I^{q,p} = Z^{q,p} \oplus Z^{q+1,p}$ and the homomorphism

$$ I^{q,p} = Z^{q,p} \oplus Z^{q+1,p} \to I^{q+1,p} = Z^{q+1,p} \oplus Z^{q+2,p}, \ (x, y) \mapsto (y, 0) $$

has kernel $Z^{q,p}$ and image $Z^{q+1,p}$ for all $p, q \geq 0$. Hence $h^q(I^{\bullet,p}) = 0$ for all $q \geq 1$. Then $0 \to I^p \to I^{0,p} \to I^{1,p} \to \cdots$ is injective resolution of the injective I^p. By Example 2.2,

$$ {}_hE_1^{p,q} = h^q(\Gamma(X, I^{\bullet,p})) = \begin{cases} \Gamma(X, I^p), & q = 0; \\ 0, & q > 0. \end{cases} $$

Then

$$ {}_hE_2^{p,q} = \begin{cases} h^p(\Gamma(X, I^\bullet)) = H^p(X, \mathscr{F}), & p = 0; \\ 0, & p > 0. \end{cases} $$

Hence ${}_hE_2^{p,q} = {}_hE_\infty^{p,q}$ and ${}_hE_2^{p,q} \Rightarrow H^{p+q}(X, \mathscr{F})$.

Finally, since two spectral sequences of the same double complex have isomorphic abutment, we have

$$ h^n(\Gamma(X, J^\bullet)) \simeq H^n(X, \mathscr{F}) $$

for all n.

□

Example 2.10. If X is an irreducible topological space with a locally constant sheaf of abelian group $\underline{A}$, then $H^n(X, \underline{A}) = 0$ for all $n \geq 1$.

Proof. Since X is irreducible, every nonempty open subset of X is connected. For all inclusion of open subsets $V \subset U$, $\underline{A}(U) \to \underline{A}(V)$ is $\mathrm{Id} : A \to A$ or $A \to 0$ or $0 \to 0$. All three possibilities are surjective. Then $\underline{A}$ is flasque. By Theorem 2.5, $H^n(X, \underline{A}) = 0$ for all $n \geq 1$. □

Remark 2.11. Sheaf cohomology is "bad", because by Example 2.10, it does not do anything even on constant sheaves. We need a better one (étale cohomology).

Despite that there are "more" flasque sheaves and even "more" acyclic sheaves, sheaf cohomology is still hard to compute. That is why we need Čech cohomology, which are easier to compute.

Definition 2.12. Let $\mathscr{U} = (U_i)_{i \in I}$ be an open covering of X whose index set I has a *well-ordering*(i.e. totally ordered such that every nonempty subset has a least element). Let $\mathscr{F}$ be a *sheaf* of $\mathcal{O}_X$-modules. If W is an open set of X with inclusion $j_W : W \hookrightarrow X$, then $(j_W)_*(\mathscr{F}|W)$ is a sheaf on X such that $(j_W)_*(\mathscr{F}|W)(U) = \mathscr{F}(U \cap W)$ for all open set $U \subset X$. Let

$$\mathscr{C}^n(\mathscr{U}, \mathscr{F}) = \prod_{i_0 < i_1 < \cdots < i_n \text{ in } I} (j_{U_{i_0} \cap \cdots \cap U_{i_n}})_*(\mathscr{F}|U_{i_0} \cap \cdots \cap U_{i_n}).$$

For all $x \in X$, define $\partial^n_x : \mathscr{C}^n(\mathscr{U}, \mathscr{F})_x \to \mathscr{C}^{n+1}(\mathscr{U}, \mathscr{F})_x$ as follows. Suppose $x \in U_{i_m}$. Let V be an open neighborhood of x in U_{i_m} and let

$$s \in \mathscr{C}^n(\mathscr{U}, \mathscr{F})(V) = \prod_{i_0 < i_1 < \cdots < i_n \text{ in } I} \Gamma(V \cap U_{i_0} \cap \cdots \cap U_{i_n}, \mathscr{F})$$

$$\partial^n_x(s)_{i_0, \cdots, i_{n+1}} = \left(\sum_{j=0}^{n} (-1)^j s_{i_0, \cdots, \widehat{i_j}, \cdots, i_{n+1}} | V \cap U_{i_0} \cap \cdots \widehat{U_{i_j}} \cdots \cap U_{i_{n+1}} \right)_x$$

We simply write

$$\partial^n_x(V, s)_{i_0 \cdots i_{n+1}} = \sum_{j=0}^{n} (-1)^j (V, s)_{i_0 \cdots \widehat{i_j} \cdots i_{n+1}}$$

Fact 2.13. $(\mathscr{C}^n, \partial^n)$ is a complex in $\mathbf{Mod}(\mathcal{O}_X)$.

Proof. Sketch: $\partial^2 = \sum_{j,k} ((-1)^{j+(k-1)} + (-1)^{j+k}) \cdots = 0$ □

Lemma 2.14.

$$0 \to \mathscr{F} \to \mathscr{C}^0(\mathscr{U}, \mathscr{F}) \xrightarrow{\partial^0} \mathscr{C}^1(\mathscr{U}, \mathscr{F}) \xrightarrow{\partial^1} \cdots$$

is exact (called the **Čech resolution** of $\mathscr{F}$).

Proof. It suffices to show that

$$0 \to \mathscr{F}_x \to \mathscr{C}^0(\mathscr{U}, \mathscr{F})_x \xrightarrow{\partial^0_x} \mathscr{C}^1(\mathscr{U}, \mathscr{F})_x \xrightarrow{\partial^1_x} \cdots$$

is exact for all $x \in X$.

Suppose $x \in U_{i_m}$, define $k^n_x : \mathscr{C}^n(\mathscr{U}, \mathscr{F})_x \to \mathscr{C}^{n-1}(\mathscr{U}, \mathscr{F})_x$: Let V be an open neighborhood of x in U_{i_m} and $s \in \mathscr{C}^n(\mathscr{U}, \mathscr{F})(V)$, let $k^n(s) \in \mathscr{C}^{n-1}(\mathscr{U}, \mathscr{F})(V)$ satisfy

$$(k^n_x(s))_{i_0, \ldots, i_{n-1}} = (s | U_{i_m} \cap U_{i_0} \cap \ldots \cap U_{i_{n-1}})_x$$

We simply write

$$k^n_x(V, s)_{i_0 \cdots i_{n-1}} = (V, s)_{i_m, i_0 \cdots i_{n-1}}$$

Then

$$\begin{aligned}
& ((\partial_x^{n-1}\circ k_x^n + k_x^{n+1}\circ\partial_x^n)(V,s))_{i_0,\dots,i_n} \\
= & \sum_{j=0}^{n}(-1)^j k_x^n(V,s)_{i_0\cdots\widehat{i_j}\cdots i_n} + \partial_x^n(V,s)_{i_m,i_0\cdots i_n} \\
= & \sum_{j=0}^{n}(-1)^j (V,s)_{i_m,i_0\cdots\widehat{i_j}\cdots i_n} + s_{i_0\cdots i_n} + \sum_{j=1}^{n+1}(-1)^j (V,s)_{i_m,i_0\cdots\widehat{i_{j-1}}\cdots i_n} \\
= & \sum_{j=0}^{n}(-1)^j (V,s)_{i_m,i_0\cdots\widehat{i_j}\cdots i_n} + s_{i_0\cdots i_n} + \sum_{\substack{l=0\\ l=j-1}}^{n}(-1)^{l+1}(V,s)_{i_m,i_0\cdots\widehat{i_k}\cdots i_n} \\
= & (V,s)_{i_0\cdots i_n}
\end{aligned}$$

Then $\partial^{n-1}\circ k^n + k^{n+1}\circ\partial^n = \mathrm{Id}$. Hence $\mathrm{Id}_{h^n(\mathscr{C}^\bullet(\mathscr{U},\mathscr{F}))} = h^n(\mathrm{Id}_{\mathscr{C}^\bullet(\mathscr{U},\mathscr{F})}) = h^n(0) = 0$. Therefore $h^n(\mathscr{C}^\bullet(\mathscr{U},\mathscr{F})) = 0$ for all $n \geq 1$. The sequence is exact at first two terms since $\mathscr{F}$ is a sheaf. □

Definition 2.15. Apply $\Gamma(X,\bullet)$ to the Čech resolution, we have a complex $\Gamma(X,\mathscr{C}^\bullet(\mathscr{U},\mathscr{F}))$. We write $\check{H}^n(\mathscr{U},\mathscr{F}) = h^n(\Gamma(X,\mathscr{C}^\bullet(\mathscr{U},\mathscr{F})))$.

3. Nov 17, Sheaf cohomology and Čech cohomology

Review: Let X be a topological space. Let $\mathcal{O}_X$ be a sheaf of rings on X.

(1) An $\mathcal{O}_X$-module $\mathscr{F}$ is acyclic if $H^n(X, \mathscr{F}) = 0$ for all $n \geq 1$. e.g. flasque sheaves of $\mathcal{O}_X$-modules; the locally constant sheaf $\underline{A}$ on an irreducible topological space X, where A is an abelian group.

(2) De Rham-Weil theorem: If $0 \to \mathscr{F} \to J^0 \to J^1 \to \cdots$ is an acyclic resolution, then $h^n(\Gamma(X, J^\bullet)) \simeq H^n(X, \mathscr{F})$.

(3) Let $\mathscr{U} = (U_i)$ be an open covering of X. Let

$$0 \to \mathscr{F} \to \mathscr{C}^0(\mathscr{U}, \mathscr{F}) \to \mathscr{C}^1(\mathscr{U}, \mathscr{F}) \to \cdots$$

be the Čech resolution. Let $C^n(\mathscr{U}, \mathscr{F}) = \Gamma(X, \mathscr{C}^n(\mathscr{U}, \mathscr{F}))$. The n-th Čech cohomology is defined to be $\check{H}^n(\mathscr{U}, \mathscr{F}) = h^n(C^\bullet(\mathscr{U}, \mathscr{F}))$

Fact 3.1. Let $\mathscr{F}$ be a sheaf on X. Let U be an open set of X. If $\mathscr{F}$ is flasque, then $\mathscr{F}|U$ is flasque.

Proof. For all inclusion of open sets $V \subset W$ in U, the restriction

$$(\mathscr{F}|U)(W) = \mathscr{F}(W) \to \mathscr{F}(V) = (\mathscr{F}|U)(V)$$

is surjective. □

Fact 3.2. Let $f : X \to Y$ be a continuous map. Let $\mathscr{F}$ be a sheaf on X. Let $f_*\mathscr{F}$ be the sheaf on Y such that $f_*\mathscr{F}(U) = F(f^{-1}(U))$ for all open set U of X. If $\mathscr{F}$ is flasque, then $f_*\mathscr{F}$ is flasque.

Proof. For all inclusion of open sets $V \subset U$ in Y, since f is continuous, $f^{-1}(V) \subset f^{-1}(U)$ is an inclusion of open sets in X. Since F is flasque, the restriction $\mathscr{F}(f^{-1}(U)) \to \mathscr{F}(f^{-1}(V))$ is surjective, i.e. $f_*\mathscr{F}(U) \to f_*\mathscr{F}(V)$ is surjective. □

Fact 3.3. If $\mathscr{F}_i$ are flasque sheaves for all $i \in I$, then $\prod_{i \in I} \mathscr{F}_i$ is a flasque sheaf.

Proof. For all inclusion of open sets $V \subset U$ in X, since $\mathscr{F}_i$ is flasque, $\mathscr{F}_i(U) \to \mathscr{F}_i(V)$ is surjective. Then $\prod_{i \in I} \mathscr{F}_i(U) \to \prod_{i \in I} \mathscr{F}_i(V)$ surjective. □

Proposition 3.4. $\check{H}^n(\mathscr{U}, \bullet) = R^n\check{H}^0(\mathscr{U}, \bullet) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Ab}$.

Proof. It suffices to show that $\check{H}^n(\mathscr{U}, \bullet)$ is a universal δ-functor. It is a δ-functor since it is cohomology of a cochain complex. We show that $\check{H}^n(\mathscr{U}, \bullet)$ is effacable. Since $\mathbf{Mod}(\mathcal{O}_X)$ has enough injectives, it suffices to show that for all injective $\mathcal{O}_X$-module I, $\check{H}^n(\mathscr{U}, I) = 0$ for all $n \geq 1$.

By Fact 3.1, Fact 3.2 and Fact 3.3,

$$\mathscr{C}^n(\mathscr{U}, \mathscr{F}) = \prod_{i_0 < i_1 < \cdots < i_n \text{ in } I} (j_{U_{i_0} \cap \cdots \cap U_{i_n}})_*(\mathscr{F}|U_{i_0} \cap \cdots \cap U_{i_n})$$

is flasque. By Theorem 2.8, $\check{H}^n(\mathscr{U}, I) = H^n(X, I)$. Finally, by Example 2.2, $H^n(X, I) = 0$. Therefore $\check{H}^n(\mathscr{U}, \bullet)$ is effacable. □

Note that $C^n(\mathscr{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_n} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_n})$, from now on, we extend our definition of Čech cohomology to

$$\check{H}^n(\mathscr{U}, \bullet) : \mathbf{PreMod}(\mathcal{O}_X) \to \mathbf{Ab}$$

Let $\iota : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{PreMod}(\mathcal{O}_X)$ be the inclusion/forgetful functor. Then $\Gamma(U, \bullet) = \Gamma_{\text{pre}}(U, \bullet) \circ \iota$ where $\Gamma_{\text{pre}}(U, \bullet)$ is the presheaf global section functor for all open set U of X,.

Lemma 3.5. Then for all $F \in \mathbf{Mod}(\mathcal{O}_X)$ and $\Gamma_{\text{pre}}(U, R^q\iota(\mathscr{F})) \simeq H^q(U, \mathscr{F})$.

Proof. Let $G_1 = \iota$, it is left exact. Let $G_2 = \Gamma_{pre}(U, \bullet) : \mathbf{PreMod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_X)$. Then G_2 is exact and hence $R^pG_2 = 0$ for all $p \geq 1$. If I is an injective $\mathcal{O}_X$-module, then $\iota(I)$ is $\Gamma_{\text{pre}}(U, \bullet)$-acyclic.

By Grothendieck spectral sequence,

$$E_2^{p,q} = R^pG_2(R^qG_1(\mathscr{F})) \Rightarrow L^{p+q} = R^{p+q}(G_2 \circ G_1)(\mathscr{F}),$$

where $E_2^{p,q} = 0$ for all $p \geq 1$. Then $E_2^{p,q} = E_\infty^{p,q} = \dfrac{F^pL^{p+q}}{F^{p+1}L^{p+q}}$. We have

$$F^1L^{p+q} = F^2L^{p+q} = \cdots = F^{p+q+1}L^{p+q} = 0$$

.

When $p = 0$, we have $\Gamma(U, R^q\iota(\mathscr{F})) = R^0G_2(R^qG_1(\mathscr{F})) = E_2^{0,q} = E_\infty^{0,q} = \dfrac{F^0L^{p+q}}{F^1L^{p+q}} = L^{p+q} = R^q(\Gamma(U, \iota(\mathscr{F}))) = R^q(\Gamma(U, \mathscr{F})) = H^q(U, \mathscr{F})$. $\square$

Lemma 3.6. Let $\mathscr{U} = (U_i)_{i \in I}$ be an open covering of X. There exists an exact sequence

$$E_2^{p,q} = \check{H}^p(\mathscr{U}, R^q\iota(\mathscr{F})) \Rightarrow H^{p+q}(X, \mathscr{F}).$$

Proof. Let $G_1 = i$ and $G_2 = \check{H}^0(\mathscr{U}, \bullet)$. If I is an injective $\mathcal{O}_X$-module, then by Proposition 3.4, $\iota(I)$ is $\check{H}^0(\mathscr{U}, \bullet)$-acyclic.

By the Grothendieck spectral sequence,

$$E_2^{p,q} = R^pG_2(R^qG_1(\mathscr{F})) \Rightarrow L^{p+q} = R^{p+q}(G_2 \circ G_1)(\mathscr{F}),$$

By Proposition 3.4, $E_2^{p,q} = R^p\check{H}^0(\mathscr{U}, \bullet)(R^q\iota(\mathscr{F})) = \check{H}^p(\mathscr{U}, R^q\iota(\mathscr{F}))$. Also

$$L^{p+q} = R^{p+q}(\check{H}^0(\mathscr{U}, \bullet) \circ \iota)(\mathscr{F}) = R^{p+q}(\Gamma(X, \bullet))(\mathscr{F}) = H^{p+q}(X, \mathscr{F}).$$

$\square$

Theorem 3.7. *If $H^q(U_i, \mathscr{F}) = 0$ for all U_i and $q \geq 1$, then $\check{H}^p(\mathscr{U}, \iota(\mathscr{F})) \simeq H^p(X, \mathscr{F})$ for all $p \geq 0$.*

Proof. By Lemma 3.5, $\Gamma_{\text{pre}}(U, R^q\iota(\mathscr{F})) \simeq H^q(U, \mathscr{F}) = 0$ for all $q \geq 1$. Then $C^\bullet(\mathscr{U}, R^q\iota(\mathscr{F})) = 0$. By Lemma 3.6,

$$E_2^{p,q} = \check{H}^p(\mathscr{U}, R^q\iota(\mathscr{F})) \Rightarrow L^{p+q} = H^{p+q}(X, \mathscr{F}).$$

Hence $E_2^{p,q} = 0$ for all $q \geq 1$. Then $0 = E_2^{p,q} = E_\infty^{p,q} = \dfrac{F^pL^{p+q}}{F^{p+1}L^{p+q}}$ for all p and for all $q \geq 1$. Then $L^p = F^0L^p = F^1L^p = \cdots = F^pL^p$.

Then $\check{H}^p(\mathscr{U}, \iota(\mathscr{F})) = E_2^{p,0} = E_\infty^{p,0} = \dfrac{F^pL^p}{F^{p+1}L^p} = L^p = H^p(X, \mathscr{F})$. $\square$

Example 3.8. $H^1(\mathbb{P}^1_{\mathbb{C}}, \underline{\mathbb{Z}}) = \mathbb{Z}$.

Proof. Let $\pi : \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{P}^1_{\mathbb{C}}$ be the canonical quotient map. Let $U_0 = \{\pi(x_0, x_1) \mid x_0 \neq 0\}$. Then $U_0 \simeq \mathbb{A}^1_{\mathbb{C}}$, $\pi(x_0, x_1) \mapsto \frac{x_1}{x_0} := x$. Let $U_1 = \{\pi(x_0, x_1) \mid x_1 \neq 0\}$. Then $U_1 \simeq \mathbb{A}^1_{\mathbb{C}}$, $\pi(x_0, x_1) \mapsto \frac{x_0}{x_1} := y$. We have $U_0 \cap U_1 = \mathbb{P}^1_{\mathbb{C}} \setminus \{\pi(1,0), \pi(0,1)\}$ has two connected components.

Since $U_0, U_1 \simeq \mathbb{A}^1_{\mathbb{C}}$ are irreducible, by Example 2.10, $H^i(U_0, \underline{Z}) = 0$ and $H^i(U_1, \underline{Z}) = 0$ for all $i \geq 1$. Let $\mathscr{U} = (U_0, U_1)$. Then

$$C^0(\mathscr{U}, \underline{Z}) = \Gamma(U_0, \underline{Z}) \oplus \Gamma(U_1, \underline{Z}) = \mathbb{Z} \oplus \mathbb{Z},$$
$$C^1(\mathscr{U}, \underline{Z}) = \Gamma(U_0 \cap U_1, \underline{Z}) = \mathbb{Z} \oplus \mathbb{Z},$$
$$d^0(a, b) = (a - b, a - b)$$

The complex is

$$0 \to C^0(\mathscr{U}, \underline{\mathbb{Z}}) \xrightarrow{d^0} C^1(\mathscr{U}, \underline{\mathbb{Z}}) \to 0$$

Finally, by Theorem 3.7,

$$H^1(\mathbb{P}^1_{\mathbb{C}}, \underline{\mathbb{Z}}) \simeq \check{H}^1(\mathscr{U}, \underline{\mathbb{Z}}) = \frac{C^1(\mathscr{U}, \underline{\mathbb{Z}})}{\mathrm{Im}(d^0)} \simeq \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z}} \simeq \mathbb{Z}.$$

□

Definition 3.9. An open covering $\mathscr{U}' = (U'_j)_{j \in J}$ is called a **refinement** of $\mathscr{U} = (U_i)_{i \in I}$ if there exists a map $\varphi : J \to I$ such that $U'_j \subset U_{\varphi(j)}$.

Fact 3.10. (1) If $\varphi : J \to I$ gives a refinement $\mathscr{U}' = (U'_j)_{j \in J}$ of $\mathscr{U} = (U_i)_{i \in I}$, then there exists a homomorphism $\varphi_* : \check{H}^n(\mathscr{U}', \mathscr{F}) \to \check{H}^n(\mathscr{U}, \mathscr{F})$.

(2) If $\varphi_1, \varphi' : J \to I$ give two refinements $\mathscr{U}' = (U'_j)_{j \in J}$ of $\mathscr{U} = (U_i)_{i \in I}$, they give the same homomorphism $\varphi_* = \varphi'_* : \check{H}^n(\mathscr{U}', \mathscr{F}) \to \check{H}^n(\mathscr{U}, \mathscr{F})$.

(3) Cohomology groups of open coverings and corresponding homomorphisms of refinements form a directed system.

Proof. Omit. □

Definition 3.11. The **Čech cohomology** of presheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules is

$$\check{H}^n(X, \mathscr{F}) = \varinjlim \check{H}^n(\mathscr{U}, \mathscr{F}).$$

Lemma 3.12. There exists an exact sequence

$$E_2^{p,q} = \check{H}^p(X, R^q\iota(\mathscr{F})) \Rightarrow H^{p+q}(X, \mathscr{F}).$$

Proof. Let $G_1 = i$ and $G_2 = \check{H}^0(X, \bullet)$. Let I be an injective sheaf of $\mathcal{O}_X$-modules. Since $\iota(I)$ is $\check{H}^0(\mathscr{U}, \bullet)$-acyclic for all $\mathscr{U}$, $\iota(I)$ is $\check{H}^0(X, \bullet)$-acyclic.

By Grothendieck spectral sequence,

$$E_2^{p,q} = R^pG_2(R^qG_1(\mathscr{F})) \Rightarrow L^{p+q} = R^{p+q}(G_2 \circ G_1)(\mathscr{F}),$$

where $E_2^{p,q} = R^p\check{H}^0(X, \bullet)(R^q\iota(\mathscr{F})) = \check{H}^p(X, R^q\iota(\mathscr{F}))$ and

$$L^{p+q} = R^{p+q}(\check{H}^0(X, \bullet) \circ \iota)(\mathscr{F}) = R^{p+q}(\Gamma(X, \bullet))(\mathscr{F}) = H^{p+q}(X, \mathscr{F}).$$

□

Definition 3.13. Let $\mathscr{F}$ be a presheaf. Its **sheafification** $\#\mathscr{F}$ is a sheaf, for all open set U, $(\#\mathscr{F})(U)$ consists of sections $s : U \to \prod_{P \in U} \mathscr{F}_P$ such that

(1) $s(P) \in \mathscr{F}_P$ for all $P \in U$.

(2) For all $P \in U$, there exists an open neighborhood V of P in U and $t \in \mathscr{F}(V)$ such that $t_Q = s(Q)$ for all $Q \in V$.

Fact 3.14. The sheafification functor $\# : \mathbf{PreMod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_X)$ is exact.

Proof. Sketch: Since $\varinjlim$ is left exact and (2) above. □

Lemma 3.15. $\check{H}^0(X, R^q\iota(\mathscr{F})) = 0$ for all $q \geq 1$.

Proof. By Fact 3.14 $R^p\# = 0$ for all $p \geq 1$. In particular, if I is an injective sheaf of $\mathcal{O}_X$-modules, then $\iota(\mathscr{F})$ is $\#$-acyclic.

By the Grothendieck spectral sequence,

$$E_2^{p,q} = R^p\#(R^q\iota(\mathscr{F})) \Rightarrow L^{p+q} = R^{p+q}(\# \circ i)(\mathscr{F}),$$

where $E_2^{p,q} = 0$ for all $p \geq 1$. Also $L^n = R^n\,\mathrm{Id}(\mathscr{F}) = 0$ for all $n \geq 1$ because Id is exact. Then $\#(R^q\iota(\mathscr{F})) = E_2^{0,q} = E_\infty^{0,q} = 0$ for all $q \geq 1$. Therefore $\check{H}^0(X, R^q\iota(\mathscr{F})) = \Gamma_{\mathrm{pre}}(X, R^q\iota(\mathscr{F})) = \Gamma(X, \#(R^q\iota(\mathscr{F}))) = 0$ for all $q \geq 1$. $\square$

Theorem 3.16. *For all $F \in \mathbf{Ab}(X)$,*

(a) $\check{H}^0(X, \iota(\mathscr{F})) \simeq H^0(X, \mathscr{F})$

(b) $\check{H}^1(X, \iota(\mathscr{F})) \simeq H^1(X, \mathscr{F})$

(c) There exists and exact sequence

$$0 \to \check{H}^2(X, \iota(\mathscr{F})) \to H^2(X, \mathscr{F}) \to \check{H}^1(X, R^1\iota(\mathscr{F})) \to \check{H}^3(X, \iota(\mathscr{F}))$$

Proof. By Lemma 3.12 and Lemma 3.15, there exists an exact sequence

$$E_2^{p,q} = \check{H}^p(X, R^q\iota(\mathscr{F})) \Rightarrow L^{p+q} = H^{p+q}(X, \mathscr{F}).$$

such that $E_2^{0,q} = 0$ for all $q \geq 1$.

(3.17)

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & & \\
 & 0 \arrow[drr] & 0 \arrow[drr] & E_2^{1,2} & E_2^{2,2} & \cdots & \\
0 \arrow[drr] & 0 \arrow[drr] & 0 \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} \arrow[drr] & E_2^{3,1} & \cdots \\
 & 0 & E_2^{0,0} \arrow[drr] & E_2^{1,0} \arrow[drr] & E_2^{2,0} \arrow[drr] & E_2^{3,0} & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & 0
\end{tikzcd}$$

(a) $\check{H}^0(X, \iota(\mathscr{F})) = E_2^{0,0} = E_\infty^{0,0} = \dfrac{F^0L^0}{F^1L^0} = L^0 = H^0(X, \mathscr{F})$.

(b) $\check{H}^1(X, \iota(\mathscr{F})) = E_2^{1,0} = E_\infty^{1,0} = \dfrac{F^1L^1}{F^2L^1} = F^1L^1$.

Since $0 = E_2^{0,1} = E_\infty^{0,1} = \dfrac{F^0L^1}{F^1L^1} = \dfrac{H^1(X, \mathscr{F})}{F^1L^1}$, we have $H^1(X, \mathscr{F}) = F^1L^1$.

Hence $\check{H}^1(X, \iota(\mathscr{F})) = F^1L^1 = H^1(X, \mathscr{F})$.

(c) $\check{H}^2(X, \iota(\mathscr{F})) = E_2^{2,0} = E_\infty^{2,0} = \dfrac{F^2L^2}{F^3L^2} = F^2L^2$.

Since $0 = E_2^{0,2} = E_\infty^{0,2} = \dfrac{F^0L^2}{F^1L^2} = \dfrac{H^2(X, \mathscr{F})}{F^1L^2}$, we have $H^2(X, \mathscr{F}) = F^1L^2$.

Since $0 = E_3^{-2,3} \to E_3^{1,1} \to E_3^{4,-1} = 0$, $E_3^{1,1} = E_\infty^{1,1} = \dfrac{F^1L^2}{F^2L^2}$.

Since $0 \to F^2L^2 \to F^1L^2 \to \dfrac{F^1L^2}{F^2L^2} \to 0$ is exact,

$$0 \to \check{H}^2(X, \iota(\mathscr{F})) \to H^2(X, \mathscr{F}) \to E_3^{1,1} \to 0$$

is exact. By $E_3^{1,1} = \ker(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0}) = \ker(\check{H}^1(X, R^1\iota(\mathscr{F})) \to \check{H}^3(X, \iota(\mathscr{F})))$, we have that

$$0 \to \check{H}^2(X, \iota(\mathscr{F})) \to H^2(X, \mathscr{F}) \to \check{H}^1(X, R^1\iota(\mathscr{F})) \to \check{H}^3(X, \iota(\mathscr{F}))$$

is exact. □

Example 3.18. Grothendieck [Gro57, 3.8.3] gives an example such that

$$\check{H}^2(X, \iota(\mathscr{F})) = 0 \text{ and } H^2(X, \mathscr{F}) = \mathbb{Z}.$$

Let $X = \mathbb{A}^2_{\mathbb{C}}$, $Y_1 = Z(x^2 + y^2 - 1)$, $Y_2 = Z(x^2 - 2x + y^2)$, $Y = Y_1 \cup Y_2$. Then Y_1 and Y_2 are irreducible and $Y_1 \cap Y_2 = \{(\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{1}{2}, -\frac{\sqrt{3}}{2})\}$ Let $\mathbb{Z}_X$ be the locally constant sheaf on X associated to $\mathbb{Z}$. If $j : F \hookrightarrow X$ is the inclusion of a closed set, then we define $\mathbb{Z}_F = j_*(\mathbb{Z}_X|F)$. If $i : U \hookrightarrow X$ is the inclusion of an open set, then we define $\mathbb{Z}_U = i_!(\mathbb{Z}_X|U)$.

(1) $H^2(X, \mathbb{Z}_{X\setminus Y}) \simeq H^1(X, \mathbb{Z}_Y)$. From the short exact sequence

$$0 \to \mathbb{Z}_{X\setminus Y} \to \mathbb{Z}_X \to \mathbb{Z}_Y \to 0,$$

we obtain a long exact sequence

$$\cdots \to H^1(X, \mathbb{Z}_X) \to H^1(X, \mathbb{Z}_Y) \to H^2(X, \mathbb{Z}_{X\setminus Y}) \to H^2(X, \mathbb{Z}_X) \to \cdots$$

Since X is irreducible, by Example 2.10, we have $H^1(X, \mathbb{Z}_X) = 0$ and $H^2(X, \mathbb{Z}_X) = 0$. Then $H^2(X, \mathbb{Z}_{X\setminus Y}) \simeq H^1(X, \mathbb{Z}_Y)$.

(2) $H^1(X, \mathbb{Z}_Y) \simeq \mathbb{Z}$. From the short exact sequence

$$0 \to \mathbb{Z}_Y \to \mathbb{Z}_{Y_1} \oplus \mathbb{Z}_{Y_2} \to \mathbb{Z}_{Y_1 \cap Y_2} \to 0$$

we obtain a long exact sequence

$$\cdots \to H^0(X, \mathbb{Z}_{Y_1} \oplus \mathbb{Z}_{Y_2}) \xrightarrow{f} H^0(X, \mathbb{Z}_{Y_1 \cap Y_2}) \to H^1(X, \mathbb{Z}_Y) \to H^1(X, \mathbb{Z}_{Y_1} \oplus \mathbb{Z}_{Y_2}) \to \cdots$$

Where $H^0(X, \mathbb{Z}_{Y_1} \oplus \mathbb{Z}_{Y_2}) \simeq H^0(X, \mathbb{Z}_{Y_1}) \oplus H^0(X, \mathbb{Z}_{Y_2}) = \Gamma(X, \mathbb{Z}_{Y_1}) \oplus \Gamma(X, \mathbb{Z}_{Y_2}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Since $Y_1 \cap Y_2$ has two connected components, $H^0(X, \mathbb{Z}_{Y_1 \cap Y_2}) = \Gamma(X, \mathbb{Z}_{Y_1 \cap Y_2}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. The map $f : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is defined as $f(m, n) = (m - n, m - n)$ for all $m, n \in \mathbb{Z}$. By Example 2.10, $H^1(X, \mathbb{Z}_{Y_1} \oplus \mathbb{Z}_{Y_2}) \simeq H^1(X, \mathbb{Z}_{Y_1}) \oplus H^1(X, \mathbb{Z}_{Y_2}) \simeq 0 \oplus 0 = 0$. Hence, $H^1(X, \mathbb{Z}_Y) = \operatorname{coker}(f) = \dfrac{\mathbb{Z} \oplus \mathbb{Z}}{\operatorname{Im}(f)} \simeq \mathbb{Z}$.

It follows from (1)(2) that $H^2(X, \mathbb{Z}_{X\setminus Y}) \simeq H^1(X, \mathbb{Z}_Y) \simeq \mathbb{Z}$.

(3) Let U be an open set such that $|U \cap Y_1 \cap Y_2| \leq 1$, we calculate $H^1(U, \mathbb{Z}_{U\setminus Y})$. From the short exact sequence

$$0 \to \mathbb{Z}_{U\setminus Y} \to \mathbb{Z}_U \to \mathbb{Z}_{U\cap Y} \to 0,$$

we obtain a long exact sequence

$$0 \to H^0(U, \mathbb{Z}_{U\setminus Y}) \to H^0(U, \mathbb{Z}_U) \xrightarrow{g} H^0(U, \mathbb{Z}_{U\cap Y}) \to H^1(U, \mathbb{Z}_{U\setminus Y}) \to H^1(U, \mathbb{Z}_U) \to \cdots$$

where $H^0(U, \mathbb{Z}_U) \simeq \mathbb{Z}$ and by Example 2.10, $H^1(U, \mathbb{Z}_U) = 0$.

(3a) Suppose $U \cap Y_1 \neq \emptyset$, $U \cap Y_2 \neq \emptyset$ and $U \cap Y_1 \cap Y_2 = \emptyset$. Then $U \cap Y$ has two connected components, $H^0(U, \mathbb{Z}_{U\cap Y}) = \mathbb{Z} \oplus \mathbb{Z}$. Also $g(m) = (m, m)$ for all $m \in \mathbb{Z}$. Hence $H^1(U, \mathbb{Z}_{U\setminus Y}) \simeq \dfrac{\mathbb{Z} \oplus \mathbb{Z}}{\operatorname{Im}(g)} \simeq \mathbb{Z}$.

(3b) Suppose $U\cap Y_1\neq\emptyset$ or $U\cap Y_2\neq\emptyset$ or $U\cap Y_1\cap Y_2\neq\emptyset$. Then $U\cap Y$ has at most one connected component. If $U\cap Y=\emptyset$, then $H^0(U,\mathbb{Z}_{U\cap Y})=0$ and hence $H^1(U,\mathbb{Z}_{U\setminus Y})=0$; If $U\cap Y$ has only one connected component, then $H^0(U,\mathbb{Z}_{U\cap Y})=\mathbb{Z}$, $g=\mathrm{Id}_{\mathbb{Z}}$ and hence $H^1(U,\mathbb{Z}_{U\setminus Y})=0$.

(4) We show that $\check{H}^1(X,R^1\iota(\mathbb{Z}_{X\setminus Y}))\simeq\mathbb{Z}$. Let $\mathscr{U}=(U_i)_{i\in I}$ be an open covering such that there exists a unique $a\in I$ such that $(\frac{1}{2},\frac{\sqrt{3}}{2})\in U_a$; there exists a unique $b\in I, b\neq a$ such that $(\frac{1}{2},-\frac{\sqrt{3}}{2})\in U_b$; for all $i\in I\setminus\{a,b\}$, $U_i\cap Y_1=\emptyset$ or $U_i\cap Y_2=\emptyset$. Then U_i satisfies (3b) for all $i\in I$.

Consider the Čech complex

$$C^0(\mathscr{U},R^1\iota(\mathbb{Z}_{X\setminus Y}))\xrightarrow{d^0}C^1(\mathscr{U},R^1\iota(\mathbb{Z}_{X\setminus Y}))\xrightarrow{d^1}C^2(\mathscr{U},R^1\iota(\mathbb{Z}_{X\setminus Y}))\to\cdots.$$

Then

$$\begin{aligned}C^0(\mathscr{U},R^1\iota(\mathbb{Z}_{X\setminus Y})) &= \prod_{i\in I}\Gamma_{\mathrm{pre}}(U_i,R^1\iota(\mathbb{Z}_{X\setminus Y}))\\ &\simeq \prod_{i\in I}H^1(U_i,\mathbb{Z}_{U_i\setminus Y}) && \text{by Lemma 3.5}\\ &= 0 && \text{by (3b)}\end{aligned}$$

.

$$\begin{aligned}C^1(\mathscr{U},R^1\iota(\mathbb{Z}_{X\setminus Y})) &= \prod_{i<j}\Gamma_{\mathrm{pre}}(U_i\cap U_j,R^1\iota(\mathbb{Z}_{X\setminus Y}))\\ &\simeq \prod_{i<j}H^1(U_i\cap U_j,\mathbb{Z}_{U_i\cap U_j\setminus Y}) && \text{by Lemma 3.5}\\ &\simeq H^1(U_a\cap U_b,\mathbb{Z}_{U_s\cap U_t\setminus Y}) && \text{by (3b)}\\ &\simeq \mathbb{Z} && \text{by (3a)}\end{aligned}$$

For all $i\in I\setminus\{a,b\}$, $U_i\cap U_a\cap U_b$ satisfies (3b), then

$$\begin{aligned}C^2(\mathscr{U},R^1\iota(\mathbb{Z}_{X\setminus Y})) &= \prod_{i<j<k}\Gamma_{\mathrm{pre}}(U_i\cap U_j\cap U_k,R^1\iota(\mathbb{Z}_{X\setminus Y}))\\ &\simeq \prod_{i<j<k}H^1(U_i\cap U_j\cap U_k,\mathbb{Z}_{U_i\cap U_j\cap U_k\setminus Y}) && \text{by Lemma 3.5}\\ &= 0 && \text{by (3b)}\end{aligned}$$

Then the the Čech complex becomes $0\xrightarrow{d^0}\mathbb{Z}\xrightarrow{d^1}0\to\cdots$ and hence $\check{H}^1(X,R^1\iota(\mathbb{Z}_{X\setminus Y}))\simeq\mathbb{Z}$.

(5) Finally, we show that $\check{H}^2(X,\iota(\mathbb{Z}_{X\setminus Y}))=0$. By Theorem 3.16(c), we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \check{H}^2(X,\iota(\mathscr{F})) \arrow[r] \arrow[d, equal] & H^2(X,\mathscr{F}) \arrow[r] \arrow[d, equal, "(1)(2)"] & \check{H}^1(X,R^1\iota(\mathscr{F})) \arrow[r] \arrow[d, equal, "(4)"] & \check{H}^3(X,\iota(\mathscr{F})) \arrow[d, equal, "\dim(X)=2"] \\ 0 \arrow[r] & \check{H}^2(X,\iota(\mathscr{F})) \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

Therefore $\check{H}^2(X,\iota(\mathbb{Z}_{X\setminus Y}))=0$.

To summarize, let $\mathscr{F}=\mathbb{Z}_{X\setminus Y}$, we have $\check{H}^2(X,\iota(\mathscr{F}))=0\neq H^2(X,\mathscr{F})=\mathbb{Z}$.

References

[Gro57] Alexander Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221. MR 0102537 ↑1, ↑17

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 ↑1

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